Optimal quadrature formulas and interpolation splines minimizing the semi-norm in the Hilbert space $K_2(P_2)$

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Abstract In this paper we construct the optimal quadrature formulas in the sense of Sard, as well as interpolation splines minimizing the semi-norm in the space $K_2(P_2)$, where $K_2(P_2)$ is a space of functions φ which φ' is absolutely continuous and φ'' belongs to $L_2(0,1)$ and $\int_0^1(\varphi''(x)+\omega^2\varphi(x))^2dx<\infty$. Optimal quadrature formulas and corresponding interpolation splines of such type are obtained by using S.L. Sobolev's method. Furthermore, order of convergence of such optimal quadrature formulas is investigted and their asymptotic optimality in the Sobolev space $L_2^{(2)}(0,1)$ is proved. These quadrature formulas and interpolation splines are exact for the trigonometric functions $\sin\omega x$ and $\cos\omega x$. Finally, a few numerical examples are included.

1 Introduction

Quadrature formulas and interpolation splines provide a basic and important tools for the numerical solution of integral and differential equations, as well as for approximation of functions in some spaces.

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This survey paper is devoted to construction of optimal quadrature formulas and interpolation splines in the space $K_2(P_2)$, which is the Hilbert space

$$K_2(P_2) := \Big\{\varphi : [0,1] \to \mathbb{R} \;\Big|\; \varphi' \text{ is absolutely continuous and } \varphi'' \in L_2(0,1)\Big\},$$

and equipped with the norm

$$\|\varphi\,|K_2(P_2)\| = \left\{\int_0^1 \left(P_2\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\varphi(x)\right)^2 \mathrm{d}x\right\}^{1/2}, \tag{1}$$

where

$$P_2\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \omega^2, \quad \omega > 0, \qquad \text{and} \qquad \int_0^1 \left(P_2\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\varphi(x)\right)^2 \mathrm{d}x < \infty.$$

The equality (1) is semi-norm and $\|\varphi\| = 0$ if and only if $\varphi(x) = c_1 \sin\omega x + c_2 \cos\omega x$.

It should be noted that for a linear differential operator of order m, $L := P_m(\mathrm{d}/\mathrm{d}x)$, Ahlberg, Nilson, and Walsh in the book [1, Chapter 6] investigated the Hilbert spaces in the context of generalized splines. Namely, with the inner product

$$\langle\varphi,\psi\rangle = \int_0^1 L\varphi(x)\cdot L\psi(x)\,\mathrm{d}x,$$

$K_2(P_m)$ is a Hilbert space if we identify functions that differ by a solution of $L\varphi = 0$. Also, such a type of spaces of periodic functions and optimal quadrature formulae were discussed in [10].

The paper is organized as follows. In Section 2 we investigate optimal quadrature formulas in the sense of Sard in $K_2(P_2)$ space. In Subsection 2.1 we give the problem of construction of optimal quadrature formulas. In subsection 2.2 we determine the extremal function which corresponds to the error functional $\ell(x)$ and give a representation of the norm of the error functional. Subsection 2.3 is devoted to a minimization of $\|\ell\|^2$ with respect to the coefficients C_ν. We obtain a system of linear equations for the coefficients of the optimal quadrature formula in the sense of Sard in the space $K_2(P_2)$. Moreover, the existence and uniqueness of the corresponding solution is proved. Explicit formulas for coefficients of the optimal quadrature formula of the form (2) are presented in Subsection 2.4. In Subsection 2.5 we give the norm of the error functional (3) of the optimal quadrature formula (2). Furthermore, we give an asymptotic analysis of this norm. Section 3 is devoted to interpolation splines minimizing the semi-norm (1) in the space $K_2(P_2)$, including an algorithm for constructing such kind of splines, as well as some numerical examples.

2 Optimal quadrature formulas in the sense of Sard

2.1 The problem of construction of optimal quadrature formulas

We consider the following quadrature formula

$$\int_0^1 \varphi(x)\,\mathrm{d}x \cong \sum_{\nu=0}^{N} C_\nu \varphi(x_\nu), \tag{2}$$

with an error functional given by

$$\ell(x) = \chi_{[0,1]}(x) - \sum_{\nu=0}^{N} C_\nu \delta(x - x_\nu), \tag{3}$$

where C_ν and x_ν $(\in [0,1])$ are coefficients and nodes of the formula (2), respectively, $\chi_{[0,1]}(x)$ is the characteristic function of the interval $[0,1]$, and $\delta(x)$ is Dirac's delta-function. We suppose that the functions $\varphi(x)$ belong to the Hilbert space $K_2(P_2)$.

The corresponding error of the quadrature formula (2) can be expressed in the form

$$R_N(\varphi) = \int_0^1 \varphi(x)\,\mathrm{d}x - \sum_{\nu=0}^{N} C_\nu \varphi(x_\nu) = (\ell,\varphi) = \int_{\mathbb{R}} \ell(x)\varphi(x)\,\mathrm{d}x \tag{4}$$

and it is a linear functional in the conjugate space $K_2^*(P_2)$ to the space $K_2(P_2)$.

By the Cauchy-Schwarz inequality

$$|(\ell,\varphi)| \le \|\varphi\,|K_2(P_2)\| \cdot \|\ell\,|K_2^*(P_2)\|$$

the error (4) can be estimated by the norm of the error functional (3), i.e.,

$$\|\ell\,|K_2^*(P_2)\| = \sup_{\|\varphi\,|K_2(P_2)\|=1} |(\ell,\varphi)|\,.$$

In this way, the error estimate of the quadrature formula (2) on the space $K_2(P_2)$ can be reduced to finding a norm of the error functional $\ell(x)$ in the conjugate space $K_2^*(P_2)$.

Obviously, this norm of the error functional $\ell(x)$ depends on the coefficients C_ν and the nodes x_ν, $\nu = 0,1,\ldots,N$. The problem of finding the minimal norm of the error functional $\ell(x)$ with respect to coefficients C_ν and nodes x_ν is called as *Nikol'skii's problem*, and the obtained formula is called *optimal quadrature formula in the sense of Nikol'skii*. This problem first considered by S.M. Nikol'skii [36], and continued by many authors (see e.g. [6, 7, 9, 10, 37, 62] and references therein). A minimization of the norm of the error functional $\ell(x)$ with respect only to coefficients C_ν, when nodes are fixed, is called as *Sard's problem*. The obtained formula is called the *optimal quadrature formula in the sense of Sard*. This problem was first investigated by A. Sard [39].

There are several methods of construction of optimal quadrature formulas in the sense of Sard (see e.g. [6, 53]). In the space $L_2^{(m)}(a,b)$, based on these methods, Sard's problem was investigated by many authors (see, for example, [4, 6, 9, 11, 12, 20, 30, 31, 33, 41, 42, 43, 45, 46, 48, 53, 54, 55, 60, 61] and references therein). Here, $L_2^{(m)}(a,b)$ is the Sobolev space of functions, with a square integrable m-th generalized derivative.

It should be noted that a construction of optimal quadrature formulas in the sense of Sard, which are exact for solutions of linear differential equations, was given in [20, 31], using the Peano kernel method, including several examples for some number of nodes.

Optimal quadrature formulas in the sense of Sard were constructed in [47] for $m = 1,2$ and in [50] for any $m \in \mathbb{N}$, using Sobolev's method in the space $W_2^{(m,m-1)}(0,1)$, with the norm defined by

$$\left\|\varphi\,|W_2^{(m,m-1)}(0,1)\right\| = \left\{\int_0^1 \left(\varphi^{(m)}(x)+\varphi^{(m-1)}(x)\right)^2 \mathrm{d}x\right\}^{1/2}.$$

In this section we give the solution of Sard's problem in the space $K_2(P_2)$, using Sobolev's method for an arbitrary number of nodes $N+1$. Namely, we find the coefficients C_ν (and the error functional $\mathring{\ell}$) such that

$$\left\|\mathring{\ell}\,|K_2^*(P_2)\right\| = \inf_{C_\nu}\left\|\ell\,|K_2^*(P_2)\right\|. \tag{5}$$

Thus, in order to construct an optimal quadrature formula in the sense of Sard in $K_2(P_2)$, we need to solve the following two problems:

PROBLEM 1. *Calculate the norm of the error functional $\ell(x)$ for the given quadrature formula* (2).

PROBLEM 2. *Find such values of the coefficients C_ν such that the equality* (5) *be satisfied with fixed nodes x_ν.*

2.2 The extremal function and representation of the norm of the error functional

To solve PROBLEM 1, i.e., to calculate the norm of the error functional (3) in the space $K_2^*(P_2)$, we use a concept of the extremal function for a given functional. The function $\psi_\ell(x)$ is called the *extremal* for the functional $\ell(x)$ (cf. [54]) if the following equality is fulfilled

$$(\ell,\psi_\ell) = \|\ell\,|K_2^*(P_2)\| \cdot \|\psi_\ell\,|K_2(P_2)\|.$$

Since $K_2(P_2)$ is a Hilbert space, the extremal function $\psi_\ell(x)$ in this space can be found using the Riesz theorem about general form of a linear continuous functional on Hilbert spaces. Then, for the functional $\ell(x)$ and for any $\varphi \in K_2(P_2)$ there exists such a function $\psi_\ell \in K_2(P_2)$, for which the following equality

$$(\ell,\varphi) = \langle \psi_\ell, \varphi \rangle \tag{6}$$

holds, where

$$\langle \psi_\ell, \varphi \rangle = \int_0^1 \left(\psi_\ell''(x) + \omega^2 \psi_\ell(x)\right)\left(\varphi''(x) + \omega^2 \varphi(x)\right) \mathrm{d}x \tag{7}$$

is an inner product defined on the space $K_2(P_2)$.

Following [25], we investigate the solution of the equation (6).

Let first $\varphi \in \mathring{C}^{(\infty)}(0,1)$, where $\mathring{C}^{(\infty)}(0,1)$ is a space of infinity-differentiable and finite functions in the interval $(0,1)$. Then from (7), an integration by parts gives

$$\langle \psi_\ell, \varphi \rangle = \int_0^1 \left(\psi_\ell^{(4)}(x) + 2\omega^2 \psi_\ell''(x) + \omega^4 \psi_\ell(x)\right)\varphi(x)\,\mathrm{d}x. \tag{8}$$

According to (6) and (8) we conclude that

$$\psi_\ell^{(4)}(x) + 2\omega^2 \psi_\ell''(x) + \omega^4 \psi_\ell(x) = \ell(x). \tag{9}$$

Thus, when $\varphi \in \mathring{C}^{(\infty)}(0,1)$ the extremal function $\psi_\ell(x)$ is a solution of the equation (9). But, we have to find the solution of (6) when $\varphi \in K_2(P_2)$.

Since the space $\mathring{C}^{(\infty)}(0,1)$ is dense in $K_2(P_2)$, then functions from $K_2(P_2)$ can be uniformly approximated as closely as desired by functions from the space $\mathring{C}^{(\infty)}(0,1)$. For $\varphi \in K_2(P_2)$ we consider the inner product $\langle \psi_\ell, \varphi \rangle$. Now, an integration by parts gives

$$\begin{aligned}\langle \psi_\ell, \varphi \rangle =& \left(\psi_\ell''(x) + \omega^2 \psi_\ell(x)\right)\varphi'(x)\,\Big|_0^1 - \left(\psi_\ell'''(x) + \omega^2 \psi_\ell'(x)\right)\varphi(x)\,\Big|_0^1 \\ &+ \int_0^1 \left(\psi_\ell^{(4)}(x) + 2\omega^2 \psi_\ell''(x) + \omega^4 \psi_\ell(x)\right)\varphi(x)\,\mathrm{d}x.\end{aligned}$$

Hence, taking into account arbitrariness $\varphi(x)$ and uniqueness of the function $\psi_\ell(x)$ (up to functions $\sin\omega x$ and $\cos\omega x$), taking into account (9), it must be fulfilled the following equation

$$\psi_\ell^{(4)}(x) + 2\omega^2 \psi_\ell''(x) + \omega^4 \psi_\ell(x) = \ell(x), \tag{10}$$

with boundary conditions

$$\psi_\ell''(0) + \omega^2 \psi_\ell(0) = 0, \quad \psi_\ell''(1) + \omega^2 \psi_\ell(1) = 0, \tag{11}$$

$$\psi_\ell'''(0) + \omega^2 \psi_\ell'(0) = 0, \quad \psi_\ell'''(1) + \omega^2 \psi_\ell'(1) = 0. \tag{12}$$

Thus, we conclude, that the extremal function $\psi_\ell(x)$ is a solution of the boundary value problem (10)–(12).

Taking the convolution of two functions f and g, i.e.,

$$(f*g)(x)=\int_{\mathbb{R}} f(x-y)g(y)\,\mathrm{d}y=\int_{\mathbb{R}} f(y)g(x-y)\,\mathrm{d}y, \tag{13}$$

we can prove the following result:

Theorem 1. *The solution of the boundary value problem* (10)–(12) *is the extremal function* $\psi_\ell(x)$ *of the error functional* $\ell(x)$ *and it has the following form*

$$\psi_\ell(x)=(G*\ell)(x)+d_1\sin\omega x+d_2\cos\omega x,$$

where d_1 *and* d_2 *are arbitrary real numbers, and*

$$G(x)=\frac{\operatorname{sign}(x)}{4\omega^3}\,(\sin\omega x-\omega x\,\cos\omega x) \tag{14}$$

is the solution of the equation

$$\psi_\ell^{(4)}(x)+2\omega^2\psi_\ell''(x)+\omega^4\psi_\ell(x)=\delta(x).$$

Proof. The general solution of a non-homogeneous differential equation can be represented as a sum of its particular solution and the general solution of the corresponding homogeneous equation. In our case, the general solution of the homogeneous equation for (10) is given by

$$\psi_\ell^h(x)=d_1\sin\omega x+d_2\cos\omega x+d_3x\,\sin\omega x+d_4x\,\cos\omega x,$$

where $d_k, k=1,2,3,4$, are arbitrary constants. It is not difficult to verify that a particular solution of the equation (10) can be expressed as a convolution of the functions $\ell(x)$ and $G(x)$ defined by (13). The function $G(x)$ is the fundamental solution of the equation (10), and it is determined by (14).

It should be noted that the following rule for finding a fundamental solution of a linear differential operator

$$P_m\left(\frac{\mathrm{d}}{\mathrm{d}x}\right):=\frac{\mathrm{d}^m}{\mathrm{d}x^m}+a_1\frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}}+\cdots+a_m,$$

where a_j are real numbers, is given in [58, pp. 88–89]. This rule needs to replace $\frac{\mathrm{d}}{\mathrm{d}x}$ by p, and then instead of the operator $P_m(\frac{\mathrm{d}}{\mathrm{d}x})$ we get the polynomial $P_m(p)$. Then we expand the expression $1/P_m(p)$ into the partial fractions, i.e.,

$$\frac{1}{P_m(p)}=\prod_j(p-\lambda)^{-k_j}=\sum_j\left[c_{j,k_j}(p-\lambda_j)^{-k_j}+\cdots+c_{j,1}(p-\lambda_j)^{-1}\right]$$

and to every partial fraction $(p-\lambda)^{-k}$ we correspond the expression $\frac{x^{k-1}\operatorname{sign}x}{2(k-1)!}\cdot e^{\lambda x}$.

Using this rule, we find the function $G(x)$ which is the fundamental solution of the operator $\frac{\mathrm{d}^4}{\mathrm{d}x^4}+2\omega^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\omega^4$ and it has the form (14).

Thus, we have the following general solution of the equation (10),

$$\psi_\ell(x)=(\ell*G)(x)+d_1\sin\omega x+d_2\cos\omega x+d_3x\ \sin\omega x+d_4x\ \cos\omega x. \tag{15}$$

In order that the function $\psi_\ell(x)$ be unique in the space $K_2(P_2)$ (up to the functions $\sin\omega x$ and $\cos\omega x$), it has to satisfy the conditions (11) and (12), where derivatives are taken in a generalized sense. In computations we need first three derivatives of the function $G(x)$:

$$\begin{aligned} G'(x) &= \frac{\operatorname{sign}x}{4\omega}x\sin\omega x,\\ G''(x) &= \frac{\operatorname{sign}x}{4\omega}(\sin\omega x+\omega x\cos\omega x),\\ G'''(x) &= \frac{\operatorname{sign}x}{4}(2\cos\omega x-\omega x\sin\omega x), \end{aligned}$$

where we used the following formulas from the theory of generalized functions [58],

$$(\operatorname{sign}x)'=2\delta(x),\quad \delta(x)f(x)=\delta(x)f(0).$$

Further, using the well-known formula

$$\frac{\mathrm{d}}{\mathrm{d}x}(f*g)(x)=(f'*g)(x)=(f*g')(x),$$

we get

$$\begin{aligned} \psi_\ell'(x) &= (\ell*G')(x)+(d_3-d_2\omega)\sin\omega x+(d_4+d_1\omega)\cos\omega x\\ &\qquad -d_4\omega x\sin\omega x+d_3\omega x\cos\omega x,\\ \psi_\ell''(x) &= (\ell*G'')(x)-(2d_4\omega+d_1\omega^2)\sin\omega x+(2d_3\omega-d_2\omega^2)\cos\omega x\\ &\qquad -d_3\omega^2x\sin\omega x-d_4\omega^2x\cos\omega x,\\ \psi_\ell'''(x) &= (\ell*G''')(x)-(3d_3\omega^2-d_2\omega^3)\sin\omega x-(3d_4\omega^2+d_1\omega^3)\cos\omega x\\ &\qquad +d_4\omega^3x\sin\omega x-d_3\omega^3x\cos\omega x. \end{aligned}$$

Now, using these expressions and (15), as well as expressions for $G^{(k)}(x)$, $k=0,1,2,3$, the boundary conditions (11) and (12) reduce to

$$\begin{cases} (\ell(y),\sin\omega y)+4d_3\omega^2=0,\\ \sin\omega\cdot(\ell(y),\cos\omega y)-\cos\omega\cdot(\ell(y),\sin\omega y)+4d_3\omega^2\cos\omega-4d_4\omega^2\sin\omega=0,\\ (\ell(y),\cos\omega y)+4d_4\omega^2=0,\\ \cos\omega\cdot(\ell(y),\cos\omega y)+\sin\omega\cdot(\ell(y),\sin\omega y)-4d_3\omega^2\sin\omega-4d_4\omega^2\cos\omega=0. \end{cases}$$

Hence, we have $d_3 = 0$, $d_4 = 0$, and therefore

$$(\ell(y), \sin \omega y) = 0, \quad (\ell(y), \cos \omega y) = 0. \tag{16}$$

Substituting these values into (15) we get the assertion of this statement. □

The equalities (16) provide that our quadrature formula is exact for functions $\sin \omega x$ and $\cos \omega x$. The case $\omega = 1$ has been recently considered in [25].

Now, using Theorem 1, we immediately obtain a representation of the norm of the error functional

$$\|\ell|K_2^*(P_2)\|^2 = (\ell, \psi_\ell) = \sum_{\nu=0}^{N}\sum_{\gamma=0}^{N} C_\nu C_\gamma G(x_\nu - x_\gamma)$$
$$- 2\sum_{\nu=0}^{N} C_\nu \int_0^1 G(x - x_\nu)\,\mathrm{d}x + \int_0^1\int_0^1 G(x-y)\,\mathrm{d}x\,\mathrm{d}y. \tag{17}$$

In the sequel we deal with PROBLEM 2.

2.3 Existence and uniqueness of optimal coefficients

Let the nodes x_ν of the quadrature formula (2) be fixed. The error functional (3) satisfies the conditions (16). Norm of the error functional $\ell(x)$ is a multidimensional function of the coefficients C_ν $(\nu = 0, 1, \ldots, N)$. For finding its minimum under the conditions (16), we apply the Lagrange method. Namely, we consider the function

$$\Psi(C_0, C_1, \ldots, C_N, d_1, d_2) = \|\ell\|^2 - 2d_1\,(\ell(x), \sin \omega x) - 2d_2\,(\ell(x), \cos \omega x)$$

and its partial derivatives equating to zero, so that we obtain the following system of linear equations

$$\sum_{\gamma=0}^{N} C_\gamma G(x_\nu - x_\gamma) + d_1 \sin \omega x_\nu + d_2 \cos \omega x_\nu = f(x_\nu), \quad \nu = 0, 1, \ldots, N, \tag{18}$$

$$\sum_{\gamma=0}^{N} C_\gamma \sin \omega x_\gamma = \frac{1 - \cos \omega}{\omega}, \quad \sum_{\gamma=0}^{N} C_\gamma \cos \omega x_\gamma = \frac{\sin \omega}{\omega}, \tag{19}$$

where $G(x)$ is determined by (14) and

$$f(x_\nu) = \int_0^1 G(x - x_\nu)\,\mathrm{d}x.$$

The system (18)–(19) has the unique solution and it gives the minimum to $\|\ell\|^2$ under the conditions (19).

The uniqueness of the solution of the system (18)–(19) is proved following [55, Chapter I]. For completeness we give it here.

First, we put $\mathbf{C}=(C_0,C_1,\dots,C_N)$ and $\mathbf{d}=(d_1,\,d_2)$ for the solution of the system of equations (18)–(19), which represents a stationary point of the function $\Psi(\mathbf{C},\mathbf{d})$.

Setting $C_\nu=\overline{C}_\nu+C_{1\nu}$, $\nu=0,1,\dots,N$, (17) and the system (18)–(19) become

$$\|\ell\|^2=\sum_{\nu=0}^{N}\sum_{\gamma=0}^{N}\overline{C}_\nu\overline{C}_\gamma G(x_\nu-x_\gamma)-2\sum_{\nu=0}^{N}(\overline{C}_\nu+C_{1\nu})\int_0^1 G(x-x_\nu)\,\mathrm{d}x$$
$$+\sum_{\nu=0}^{N}\sum_{\gamma=0}^{N}\left(2\overline{C}_\nu C_{1\gamma}+C_{1\nu}C_{1\gamma}\right)G(x_\nu-x_\gamma)+\int_0^1\int_0^1 G(x-y)\,\mathrm{d}x\,\mathrm{d}y \qquad (20)$$

and

$$\sum_{\gamma=0}^{N}\overline{C}_\gamma G(x_\nu-x_\gamma)+d_1\sin\omega x_\nu+d_2\cos\omega x_\nu=F(x_\nu),\quad \nu=0,1,\dots,N, \qquad (21)$$

$$\sum_{\gamma=0}^{N}\overline{C}_\gamma\sin\omega x_\gamma=0,\quad \sum_{\gamma=0}^{N}\overline{C}_\gamma\cos\omega x_\gamma=0, \qquad (22)$$

respectively, where

$$F(x_\nu)=f(x_\nu)-\sum_{\gamma=0}^{N}C_{1\gamma}G(x_\nu-x_\gamma)$$

and $C_{1\gamma}$, $\gamma=0,1,\dots,N$, are particular solutions of the system (19).

Hence, we directly get, that the minimization of (17) under the conditions (16) by C_ν is equivalent to the minimization of the expression (20) by $\overline{C}_\nu$ under the conditions (22). Therefore, it is sufficient to prove that the system (21)–(22) has the unique solution with respect to $\overline{\mathbf{C}}=(\overline{C}_0,\overline{C}_1,\dots,\overline{C}_N)$ and $\mathbf{d}=(d_1,\,d_2)$ and this solution gives the conditional minimum for $\|\ell\|^2$. From the theory of the conditional extremum, we need the positivity of the quadratic form

$$\Phi(\overline{\mathbf{C}})=\sum_{\nu=0}^{N}\sum_{\gamma=0}^{N}\frac{\partial^2\Psi}{\partial\overline{C}_\nu\partial\overline{C}_\gamma}\overline{C}_\nu\overline{C}_\gamma \qquad (23)$$

on the set of vectors $\overline{\mathbf{C}}=(\overline{C}_0,\overline{C}_1,\dots,\overline{C}_N)$, under the condition

$$S\overline{\mathbf{C}}=0, \qquad (24)$$

where S is the matrix of the system of equations (22),

$$S=\begin{pmatrix}\sin\omega x_0 & \sin\omega x_1 & \cdots & \sin\omega x_N\\ \cos\omega x_0 & \cos\omega x_1 & \cdots & \cos\omega x_N\end{pmatrix}.$$

Now, we show, that in this case the condition is satisfied.

Theorem 2. *For any nonzero vector $\overline{\mathbf{C}} \in \mathbb{R}^{N+1}$, lying in the subspace $S\overline{\mathbf{C}} = 0$, the function $\Phi(\overline{\mathbf{C}})$ is strictly positive.*

Proof. Using the definition of the function $\Psi(\mathbf{C}, \mathbf{d})$ and the previous equations, (23) reduces to

$$\Phi\left(\overline{\mathbf{C}}\right) = 2\sum_{\nu=0}^{N}\sum_{\gamma=0}^{N} G(x_\nu - x_\gamma)\overline{C}_\nu \overline{C}_\gamma. \tag{25}$$

Consider now a linear combination of shifted delta functions

$$\delta_{\overline{\mathbf{C}}}(x) = \sqrt{2}\sum_{\nu=0}^{N} \overline{C}_\nu \delta(x - x_\nu). \tag{26}$$

By virtue of the condition (24), this functional belongs to the space $K_2^*(P_2)$. So, it has an extremal function $u_{\overline{\mathbf{C}}}(x) \in K_2(P_2)$, which is a solution of the equation

$$\left(\frac{\mathrm{d}^4}{\mathrm{d}x^4} + 2\omega^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \omega^4\right) u_{\overline{\mathbf{C}}}(x) = \delta_{\overline{\mathbf{C}}}(x). \tag{27}$$

As $u_{\overline{\mathbf{C}}}(x)$ we can take a linear combination of shifts of the fundamental solution $G(x)$:

$$u_{\overline{\mathbf{C}}}(x) = \sqrt{2}\sum_{\nu=0}^{N} \overline{C}_\nu G(x - x_\nu),$$

and we can see that

$$\left\|u_{\overline{\mathbf{C}}}|K_2(P_2)\right\|^2 = \left(\delta_{\overline{\mathbf{C}}}, u_{\overline{\mathbf{C}}}\right) = 2\sum_{\nu=0}^{N}\sum_{\gamma=0}^{N} \overline{C}_\nu \overline{C}_\gamma G(x_\nu - x_\gamma) = \Phi\left(\overline{\mathbf{C}}\right).$$

Thus, it is clear that for a nonzero $\overline{\mathbf{C}}$ the function $\Phi\left(\overline{\mathbf{C}}\right)$ is strictly positive and Theorem 2 is proved. □

If the nodes $x_0, x_1, \ldots, x_N$ are selected such that the matrix S has the right inverse, then the system of equations (21)–(22) has the unique solution, as well as the system of equations (18)–(19).

Theorem 3. *If the matrix S has the right inverse matrix, then the main matrix Q of the system of equations* (21)–(22) *is nonsingular.*

Proof. We denote by M the matrix of the quadratic form $\frac{1}{2}\Phi\left(\overline{\mathbf{C}}\right)$, given in (25). It is enough to consider the homogenous system of linear equations

$$Q\begin{pmatrix}\overline{\mathbf{C}}\\ \mathbf{d}\end{pmatrix} = \begin{pmatrix} M & S^* \\ S & 0\end{pmatrix}\begin{pmatrix}\overline{\mathbf{C}}\\ \mathbf{d}\end{pmatrix} = 0 \tag{28}$$

and prove that it has only the trivial solution.

Let $\overline{\mathbf{C}}$, $\mathbf{d}$ be a solution of (28). Consider the function $\delta_{\overline{\mathbf{C}}}(x)$, defined before by (26). As an extremal function for $\delta_{\overline{\mathbf{C}}}(x)$ we take the following function

$$u_{\overline{\mathbf{C}}}(x)=\sqrt{2}\sum_{\nu=0}^{N}\overline{C}_\nu G(x-x_\nu)+d_1\sin\omega x+d_2\cos\omega x.$$

This is possible, because $u_{\overline{\mathbf{C}}}$ belongs to the space $K_2(P_2)$ and it is a solution of the equation (27). The first $N+1$ equations of the system (28) mean that $u_{\overline{\mathbf{C}}}(x)$ takes the value zero at all nodes x_ν, $\nu=0,1,\dots,N$. Then, for the norm of the functional $\delta_{\overline{\mathbf{C}}}(x)$ in $K_2^*(P_2)$, we have

$$\left\|\delta_{\overline{\mathbf{C}}}|K_2^*(P_2)\right\|^2=\left(\delta_{\overline{\mathbf{C}}},u_{\overline{\mathbf{C}}}\right)=\sqrt{2}\sum_{\nu=0}^{N}\overline{C}_\nu\,u_{\overline{\mathbf{C}}}(x_\nu)=0,$$

which is possible only when $\overline{\mathbf{C}}=0$. According to this fact, from the first $N+1$ equations of the system (28) we obtain $S^*\mathbf{d}=0$. Since the matrix S is a right-inversive (by the hypotheses of this theorem), we conclude that S^* has the left inverse matrix, and therefore $\mathbf{d}=0$, i.e., $d_1=d_2=0$, which completes the proof. □

According to (17) and Theorems 2 and 3, it follows that at fixed values of the nodes x_ν, $\nu=0,1,\dots,N$, the norm of the error functional $\ell(x)$ has the unique minimum for some concrete values of $C_\nu=\overset{\circ}{C}_\nu$, $\nu=0,1,\dots,N$. As we mentioned in the first section, the quadrature formula with such coefficients $\overset{\circ}{C}_\nu$ is called *the optimal quadrature formula in the sense of Sard*, and $\overset{\circ}{C}_\nu$, $\nu=0,1,\dots,N$, are the *optimal coefficients*. In the sequel, for convenience the optimal coefficients $\overset{\circ}{C}_\nu$ will be denoted only as C_ν.

2.4 Coefficients of optimal quadrature formula

In this subsection we solve the system (18)–(19) and find an explicit formula for the coefficients C_ν. We use a similar method, offered by Sobolev [53, 55] for finding optimal coefficients in the space $L_2^{(m)}(0,1)$. Here, we mainly use a concept of functions of a discrete argument and the corresponding operations (see [54] and [55]). For completeness we give some of definitions.

Let nodes x_ν are equal spaced, i.e., $x_\nu=\nu h$, $h=1/N$. Assume that $\varphi(x)$ and $\psi(x)$ are real-valued functions defined on the real line $\mathbb{R}$.

Definition 1. The function $\varphi(h\nu)$ is a *function of discrete argument* if it is given on some set of integer values of ν.

Definition 2. The *inner product* of two discrete functions $\varphi(h\nu)$ and $\psi(h\nu)$ is given by

$$[\varphi,\psi]=\sum_{\nu=-\infty}^{\infty}\varphi(h\nu)\cdot\psi(h\nu),$$

if the series on the right hand side converges absolutely.

Definition 3. The *convolution* of two functions $\varphi(h\nu)$ and $\psi(h\nu)$ is the inner product

$$\varphi(h\nu)*\psi(h\nu)=[\varphi(h\gamma),\psi(h\nu-h\gamma)]=\sum_{\gamma=-\infty}^{\infty}\varphi(h\gamma)\cdot\psi(h\nu-h\gamma).$$

Suppose that $C_\nu=0$ when $\nu<0$ and $\nu>N$. Using these definitions, the system (18)–(19) can be rewritten in the convolution form

$$G(h\nu)*C_\nu+d_1\sin(h\omega\nu)+d_2\cos(h\omega\nu)=f(h\nu),\quad \nu=0,1,\ldots,N,\quad (29)$$

$$\sum_{\nu=0}^{N}C_\nu\sin(h\omega\nu)=\frac{1-\cos\omega}{\omega},\quad \sum_{\nu=0}^{N}C_\nu\cos(h\omega\nu)=\frac{\sin\omega}{\omega},\quad (30)$$

where

$$f(h\nu)=\frac{1}{4\omega^4}\big[4-(2+2\cos\omega+\omega\sin\omega)\cos(h\omega\nu)-(2\sin\omega-\omega\cos\omega)\sin(h\omega\nu)$$
$$+\sin\omega\cdot(h\omega\nu)\cos(h\omega\nu)-(1+\cos\omega)\cdot(h\omega\nu)\sin(h\omega\nu)\big].\quad (31)$$

Now, we consider the following problem:

PROBLEM 3. *For a given $f(h\nu)$ find a discrete function C_ν and unknown coefficients d_1,d_2, which satisfy the system* (29) − (30).

Further, instead of C_ν we introduce the functions $v(h\nu)$ and $u(h\nu)$ by

$$v(h\nu)=G(h\nu)*C_\nu\quad\text{and}\quad u(h\nu)=v(h\nu)+d_1\sin(h\omega\nu)+d_2\cos(h\omega\nu).$$

In this statement it is necessary to express C_ν by the function $u(h\nu)$. For this we have to construct such an operator $D(h\nu)$, which satisfies the equation

$$D(h\nu)*G(h\nu)=\delta(h\nu),\quad (32)$$

where $\delta(h\nu)$ is equal to 0 when $\nu\neq 0$ and is equal to 1 when $\nu=0$, i.e., $\delta(h\nu)$ is a discrete delta-function.

In connection with this, a discrete analogue $D(h\nu)$ of the differential operator

$$\frac{\mathrm{d}^4}{\mathrm{d}x^4}+2\omega^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\omega^4,\quad (33)$$

which satisfies (32) was constructed in [24] and some properties were investigated.

Following [24] we have:

Theorem 4. *The discrete analogue of the differential operator* (33) *satisfying the equation* (32) *has the form*

$$D(h\nu)=p\begin{cases} A_1\lambda_1^{|\nu|-1}, & |\nu|\geq 2,\\ 1+A_1, & |\nu|=1,\\ C+\dfrac{A_1}{\lambda_1}, & \nu=0, \end{cases} \tag{34}$$

where $p=2\omega^3/(\sin h\omega-h\omega\cos h\omega)$,

$$A_1=\frac{(2h\omega)^2\sin^4(h\omega)\lambda_1^2}{(\lambda_1^2-1)(\sin h\omega-h\omega\cos h\omega)^2},\quad C=\frac{2h\omega\cos(2h\omega)-\sin(2h\omega)}{\sin h\omega-h\omega\cos h\omega}, \tag{35}$$

and

$$\lambda_1=\frac{2h\omega-\sin(2h\omega)-2\sin(h\omega)\sqrt{h^2\omega^2-\sin^2(h\omega)}}{2(h\omega\cos(h\omega)-\sin(h\omega))} \tag{36}$$

is a zero of the polynomial

$$Q_2(\lambda)=\lambda^2+\frac{2h\omega-\sin(2h\omega)}{\sin h\omega-h\omega\cos h\omega}\lambda+1, \tag{37}$$

and $|\lambda_1|<1$, *h is a small parameter,* $\omega>0$, $|h\omega|<1$.

Theorem 5. *The discrete analogue* $D(h\nu)$ *of the differential operator* (33) *satisfies the following equalities:*

1) $D(h\nu)*\sin(h\omega\nu)=0$,
2) $D(h\nu)*\cos(h\omega\nu)=0$,
3) $D(h\nu)*(h\omega\nu)\sin(h\omega\nu)=0$,
4) $D(h\nu)*(h\omega\nu)\cos(h\omega\nu)=0$,
5) $D(h\nu)*G(h\nu)=\delta(h\nu)$.

Here $G(h\nu)$ *is the function of discrete argument, corresponding to the function* $G(x)$ *defined by* (14)*, and* $\delta(h\nu)$ *is the discrete delta-function.*

Then, taking into account (32) and Theorems 4 and 5, for optimal coefficients we have

$$C_\nu=D(h\nu)*u(h\nu). \tag{38}$$

Thus, if we find the function $u(h\nu)$, then the optimal coefficients can be obtained from (38). In order to calculate the convolution (38) we need a representation of the function $u(h\nu)$ for all integer values of ν. According to (29) we get that $u(h\nu)=f(h\nu)$ when $h\nu\in[0,1]$. Now, we need a representation of the function $u(h\nu)$ when $\nu<0$ and $\nu>N$.

Since $C_\nu=0$ for $h\nu\notin[0,1]$, then $C_\nu=D(h\nu)*u(h\nu)=0$, $h\nu\notin[0,1]$. Now, we calculate the convolution $v(h\nu)=G(h\nu)*C_\nu$ when $h\nu\notin[0,1]$.

Let $\nu<0$, then, taking into account equalities (14) and (30), we have

$$
\begin{aligned}
v(h\nu) &= G(h\nu) * C_\nu = \sum_{\gamma=-\infty}^{\infty} C_\gamma G(h\nu - h\gamma) \\
&= \sum_{\gamma=0}^{N} C_\gamma \frac{\operatorname{sign}(h\nu - h\gamma)}{4\omega^3} \big(\sin(h\omega\nu - h\omega\gamma) - (h\omega\nu - h\omega\gamma)\cos(h\omega\nu - h\omega\gamma)\big) \\
&= -\frac{1}{4\omega^3}\Bigg[(\sin(h\omega\nu) - h\omega\nu\cos(h\omega\nu))\frac{\sin\omega}{\omega} \\
&\quad -(\cos(h\omega\nu) + h\omega\nu\sin(h\omega\nu))\frac{(1-\cos\omega)}{\omega} \\
&\quad +\cos(h\omega\nu)\sum_{\gamma=0}^{N} C_\gamma h\omega\gamma\cos(h\omega\gamma) + \sin(h\omega\nu)\sum_{\gamma=0}^{N} C_\gamma h\omega\gamma\sin(h\omega\gamma)\Bigg].
\end{aligned}
$$

Denoting

$$
b_1 = \frac{1}{4\omega^3}\sum_{\gamma=0}^{N} C_\gamma h\omega\gamma\sin(h\omega\gamma) \quad \text{and} \quad b_2 = \frac{1}{4\omega^3}\sum_{\gamma=0}^{N} C_\gamma h\omega\gamma\cos(h\omega\gamma),
$$

we get for $\nu < 0$

$$
\begin{aligned}
v(h\nu) &= -\frac{1}{4\omega^3}\Bigg[(\sin(h\omega\nu) - h\omega\nu\cos(h\omega\nu))\frac{\sin\omega}{\omega} - (\cos(h\omega\nu) \\
&\quad + h\omega\nu\sin(h\omega\nu))\frac{(1-\cos\omega)}{\omega} + 4\omega^3 b_1\sin(h\omega\nu) + 4\omega^3 b_2\cos(h\omega\nu)\Bigg],
\end{aligned}
$$

and for $\nu > N$

$$
\begin{aligned}
v(h\nu) &= \frac{1}{4\omega^3}\Bigg[(\sin(h\omega\nu) - h\omega\nu\cos(h\omega\nu))\frac{\sin\omega}{\omega} - (\cos(h\omega\nu) \\
&\quad + h\omega\nu\sin(h\omega\nu))\frac{(1-\cos\omega)}{\omega} + 4\omega^3 b_1\sin(h\omega\nu) + 4\omega^3 b_2\cos(h\omega\nu)\Bigg].
\end{aligned}
$$

Now, setting

$$
d_1^- = d_1 - b_1, \quad d_2^- = d_2 - b_2, \quad d_1^+ = d_1 + b_1, \quad d_2^+ = d_2 + b_2
$$

we formulate the following problem:

PROBLEM 4. *Find the solution of the equation*

$$
D(h\nu) * u(h\nu) = 0, \quad h\nu \notin [0,1], \tag{39}
$$

in the form

$$u(h\nu)=\begin{cases}-\frac{\sin\omega}{4\omega^4}(\sin(h\omega\nu)-h\omega\nu\cos(h\omega\nu))+\frac{1-\cos\omega}{4\omega^4}(\cos(h\omega\nu)\\ \qquad +h\omega\nu\sin(h\omega\nu))+d_1^-\sin(h\omega\nu)+d_2^-\cos(h\omega\nu), & \nu<0,\\ f(h\nu), & 0\le\nu\le N,\\ \frac{\sin\omega}{4\omega^4}(\sin(h\omega\nu)-h\omega\nu\cos(h\omega\nu))-\frac{1-\cos\omega}{4\omega^4}(\cos(h\omega\nu)\\ \qquad +h\omega\nu\sin(h\omega\nu))+d_1^+\sin(h\omega\nu)+d_2^+\cos(h\omega\nu), & \nu>N,\end{cases} \tag{40}$$

where d_1^-, d_2^-, d_1^+, d_2^+ are unknown coefficients.

It is clear that

$$d_1=\frac{1}{2}\left(d_1^++d_1^-\right),\ \ b_1=\frac{1}{2}\left(d_1^+-d_1^-\right),\ \ d_2=\frac{1}{2}\left(d_2^++d_2^-\right),\ \ b_2=\frac{1}{2}\left(d_2^+-d_2^-\right).$$

These unknowns d_1^-, d_2^-, d_1^+, d_2^+ can be found from the equation (39), using the function $D(h\nu)$. Then, the explicit form of the function $u(h\nu)$ and optimal coefficients C_ν can be obtained. Thus, in this way PROBLEM 4, as well as PROBLEM 3, can be solved.

However, instead of this, using $D(h\nu)$ and $u(h\nu)$ and taking into account (38), we find here expressions for the optimal coefficients C_ν, $\nu=1,\dots,N-1$. For this purpose we introduce the following notations

$$m=\sum_{\gamma=1}^{\infty}\frac{A_1p}{\lambda_1}\lambda_1^{\gamma}\Bigg[-\frac{\sin\omega}{4\omega^4}(\sin(-h\omega\gamma)+h\omega\gamma\cos(h\omega\gamma))-f(-h\gamma)$$
$$+\ \frac{1-\cos\omega}{4\omega^4}(\cos(h\omega\gamma)+h\omega\gamma\sin(h\omega\gamma))-d_1^-\sin(h\omega\gamma)+d_2^-\cos(h\omega\gamma)\Bigg],$$
$$n=\sum_{\gamma=1}^{\infty}\frac{A_1p}{\lambda_1}\lambda_1^{\gamma}\Bigg[\frac{\sin\omega}{4\omega^4}(\sin((N+\gamma)h\omega)-(N+\gamma)h\omega\cos((N+\gamma)h\omega))-f((N+\gamma)h)$$
$$-\ \frac{1-\cos\omega}{4\omega^4}(\cos((N+\gamma)h\omega)+(N+\gamma)h\omega\sin((N+\gamma)h\omega))+d_1^+\sin((N+\gamma)h\omega)$$
$$+\ d_2^+\cos((N+\gamma)h\omega)\Bigg].$$

The series in the previous expressions are convergent, because $|\lambda_1|<1$.

Theorem 6. *The coefficients of optimal quadrature formulas in the sense of Sard of the form* (2) *in the space $K_2(P_2)$ have the following representation*

$$C_\nu=\frac{4(1-\cos h\omega)}{\omega\cdot(h\omega+\sin h\omega)}+m\lambda_1^{\nu}+n\lambda_1^{N-\nu},\quad \nu=1,\dots,N-1, \tag{41}$$

where m and n are defined above, and λ_1 is given in Theorem 4.

Proof. Let $\nu\in\{1,\dots,N-1\}$. Then from (38), using (34) and (40), we have

$$
\begin{aligned}
C_\nu &= D(h\nu) * u(h\nu) = \sum_{\gamma=-\infty}^{\infty} D(h\nu - h\gamma)u(h\gamma) \\
&= \sum_{\gamma=-\infty}^{-1} D(h\nu - h\gamma)u(h\gamma) + \sum_{\gamma=0}^{N} D(h\nu - h\gamma)u(h\gamma) + \sum_{\gamma=N+1}^{\infty} D(h\nu - h\gamma)u(h\gamma) \\
&= D(h\nu) * f(h\nu) + \sum_{\gamma=1}^{\infty} \frac{A_1 p}{\lambda_1}\lambda_1^{\nu+\gamma}\Big[-\frac{\sin\omega}{4\omega^4}(\sin(-h\omega\gamma) + h\omega\gamma\cos(h\omega\gamma)) \\
&+ \frac{1-\cos\omega}{4\omega^4}(\cos(h\omega\gamma) + h\omega\gamma\sin(h\omega\gamma)) - d_1^- \sin(h\omega\gamma) + d_2^- \cos(h\omega\gamma) - f(-h\gamma)\Big] \\
&+ \sum_{\gamma=1}^{\infty} \frac{A_1 p}{\lambda_1}\lambda_1^{N+\gamma-\nu}\Big[\frac{\sin\omega}{4\omega^4}(\sin((N+\gamma)h\omega) - (N+\gamma)h\omega\cos((N+\gamma)h\omega)) \\
&- \frac{1-\cos\omega}{4\omega^4}(\cos((N+\gamma)h\omega) + (N+\gamma)h\omega\sin((N+\gamma)h\omega)) \\
&+ d_1^+ \sin((N+\gamma)h\omega) + d_2^+ \cos((N+\gamma)h\omega) - f((N+\gamma)h)\Big].
\end{aligned}
$$

Hence, taking into account the previous notations, we get

$$
C_\nu = D(h\nu) * f(h\nu) + m\lambda_1^\nu + n\lambda_1^{N-\nu}. \tag{42}
$$

Now, using Theorems 4 and 5 and equality (31), we calculate the convolution $D(h\nu) * f(h\nu)$,

$$
\begin{aligned}
D(h\nu) * f(h\nu) &= D(h\nu) * \frac{1}{\omega^4} = \frac{1}{\omega^4}\sum_{\gamma=-\infty}^{\infty} D(h\gamma) \\
&= \frac{1}{\omega^4}\left(D(0) + 2D(h) + 2\sum_{\gamma=2}^{\infty} D(h\gamma)\right) \\
&= \frac{4(1-\cos h\omega)}{\omega\cdot(h\omega + \sin h\omega)}.
\end{aligned}
$$

Substituting this convolution into (42) we obtain (41), and Theorem 6 is proved. □

According Theorem 6 it is clear, that in order to obtain the exact expressions of the optimal coefficients C_ν we need only m and n. They can be found from an identity with respect to $(h\nu)$, which can be obtained by substituting the equality (41) into (29). Namely, equating the corresponding coefficients the left and the right hand sides of the equation (29) we find m and n. The coefficients C_0 and C_N follow directly from (30).

Now we can formulate and prove the following result:

Theorem 7. *The coefficients of the optimal quadrature formulas in the sense of Sard of the form* (2) *in the space* $K_2(P_2)$ *are*

$$C_\nu = \begin{cases} \dfrac{2\sin h\omega-(h\omega+\sin h\omega)\cos h\omega}{(h\omega+\sin h\omega)\omega\sin h\omega}+\dfrac{(h\omega-\sin h\omega)(\lambda_1+\lambda_1^{N-1})}{(h\omega+\sin h\omega)\omega\sin h\omega\cdot(1+\lambda_1^N)}, \\ \nu=0,N, \\ \\ \dfrac{4(1-\cos h\omega)}{\omega(h\omega+\sin h\omega)}+\dfrac{2h(h\omega-\sin h\omega)\sin h\omega\left(\lambda_1^\nu+\lambda_1^{N-\nu}\right)}{(h\omega+\sin h\omega)(h\omega\cos h\omega-\sin h\omega)(1+\lambda_1^N)}, \\ \nu=1,\dots,N-1, \end{cases}$$

where λ_1 *is given in Theorem* 4 *and* $|\lambda_1|<1$.

Proof. First from equations (30) we have

$$C_0 = \frac{\sin\omega}{\omega}-\frac{\cos\omega(1-\cos\omega)}{\omega\sin\omega}-\sum_{\gamma=1}^{N-1}C_\gamma\cos(h\omega\gamma)+\frac{\cos\omega}{\sin\omega}\sum_{\gamma=1}^{N-1}C_\gamma\sin(h\omega\gamma),$$
$$C_N = \frac{1-\cos\omega}{\omega\sin\omega}-\frac{1}{\sin\omega}\sum_{\gamma=1}^{N-1}C_\gamma\sin(h\omega\gamma).$$

Hence, using (41), after some simplifications we get

$$\begin{aligned} C_0 = {} & \frac{(h\omega-\sin h\omega)(1-\cos\omega)+2\sin\omega\cdot(1-\cos h\omega)}{\omega\sin\omega\cdot(h\omega+\sin h\omega)} \\ & -m\,\frac{\lambda_1(\sin\omega\ \cos h\omega-\cos\omega\ \sin h\omega)+\lambda_1^{N+1}\sin h\omega-\lambda_1^2\sin\omega}{(\lambda_1^2+1-2\lambda_1\cos h\omega)\ \sin\omega} \\ & -n\,\frac{\lambda_1^{N+1}(\sin\omega\ \cos h\omega-\sin h\omega\ \cos\omega)+\lambda_1\sin h\omega-\lambda_1^N\sin\omega}{(\lambda_1^2+1-2\lambda_1\cos h\omega)\ \sin\omega}, \end{aligned}$$

$$\begin{aligned} C_N = {} & \frac{(h\omega-\sin h\omega)(1-\cos\omega)+2\sin\omega\cdot(1-\cos h\omega)}{\omega\sin\omega\cdot(h\omega+\sin h\omega)} \\ & -m\,\frac{\lambda_1^{N+1}(\sin\omega\ \cos h\omega-\sin h\omega\ \cos\omega)+\lambda_1\sin h\omega-\lambda_1^N\sin\omega}{(\lambda_1^2+1-2\lambda_1\cos h\omega)\ \sin\omega} \\ & -n\,\frac{\lambda_1(\sin\omega\ \cos h\omega-\cos\omega\ \sin h\omega)+\lambda_1^{N+1}\sin h\omega-\lambda_1^2\sin\omega}{(\lambda_1^2+1-2\lambda_1\cos h\omega)\ \sin\omega}. \end{aligned}$$

Further, we consider the convolution $G(h\nu)*C_\nu$ in equation (29), i.e.,

$$
\begin{aligned}
G(h\nu)*C_\nu &= \sum_{\gamma=0}^{N} C_\gamma G(h\nu-h\gamma)\\
&= \sum_{\gamma=0}^{N} C_\gamma \frac{\operatorname{sign}(h\nu-h\gamma)}{4\omega^3}\,[\sin(h\omega\nu-h\omega\gamma)\\
&\qquad -(h\omega\nu-h\omega\gamma)\cos(h\omega\nu-h\omega\gamma)]\\
&= S_1-S_2, \qquad (43)
\end{aligned}
$$

where

$$
S_1=\frac{1}{2\omega^3}\sum_{\gamma=0}^{\nu} C_\gamma\,[\sin(h\omega\nu-h\omega\gamma)-(h\omega\nu-h\omega\gamma)\cos(h\omega\nu-h\omega\gamma)]
$$

and

$$
S_2=\frac{1}{4\omega^3}\sum_{\gamma=0}^{N} C_\gamma\,[\sin(h\omega\nu-h\omega\gamma)-(h\omega\nu-h\omega\gamma)\cos(h\omega\nu-h\omega\gamma)]\,.
$$

Using (41) we obtain

$$
\begin{aligned}
S_1 &= \frac{1}{2\omega^3}C_0[\sin(h\omega\nu)-h\omega\nu\cos(h\omega\nu)]\\
&\quad +\frac{1}{2\omega^3}\sum_{\gamma=0}^{\nu-1}(k+m\lambda_1^{\nu-\gamma}+n\lambda_1^{N+\gamma-\nu})[\sin(h\omega\gamma)-h\omega\gamma\cos(h\omega\gamma)],
\end{aligned}
$$

where $k=4(1-\cos h\omega)/(\omega(h\omega+\sin h\omega))$. After some calculations and simplifications S_1 can be reduced to the following form

$$
\begin{aligned}
S_1 &= \frac{1}{\omega^4}\big[1-\cos(h\nu)\big]+\Bigg[\frac{(h\omega-\sin h\omega)(1-\cos\omega)}{2\omega^4\sin\omega\,(h\omega+\sin h\omega)}\\
&+\frac{m}{2\omega^3}\left(\frac{(\lambda_1\cos\omega-\lambda_1^{N+1})\sin h\omega}{\sin\omega\,(\lambda_1^2+1-2\lambda_1\cos h\omega)}+\frac{(\lambda_1-\lambda_1^3)\,h\omega\sin h\omega}{(\lambda_1^2+1-2\lambda_1\cos h\omega)^2}\right)\\
&+\frac{n}{2\omega^3}\left(\frac{(\lambda_1^{N+1}\cos\omega-\lambda_1)\sin h\omega}{\sin\omega\,(\lambda_1^2+1-2\lambda_1\cos h\omega)}+\frac{(\lambda_1^{N+3}-\lambda_1^{N+1})h\omega\sin h\omega}{(\lambda_1^2+1-2\lambda_1\cos h\omega)^2}\right)\Bigg]\sin(h\omega\nu)\\
&-\left[\frac{\sin h\omega}{\omega^4\,(h\omega+\sin h\omega)}+\frac{\lambda_1(m+n\lambda_1^N)\sin h\omega}{2\omega^3(\lambda_1^2+1-2\lambda_1\cos h\omega)}\right](h\omega\nu)\sin(h\omega\nu)\\
&+\left[\frac{(h\omega-\sin h\omega)(\cos\omega-1)}{\omega\,(h\omega+\sin h\omega)}-\frac{\lambda_1[(m+n\lambda_1^N)\cos\omega-(n+m\lambda_1^N)]\sin h\omega}{\lambda_1^2+1-2\lambda_1\cos h\omega}\right]\\
&\qquad\qquad\times\frac{h\omega\nu\cos(h\omega\nu)}{2\omega^3\sin\omega},
\end{aligned}
$$

where we used the fact that λ_1 is a zero of the polynomial $Q_2(\lambda)$ defined by (37).

Now, keeping in mind (30), for S_2 we get the following expression

$$S_2 = \frac{1}{4\omega^3}\left[\frac{\sin\omega}{\omega}\,\sin(h\omega\nu) - \frac{1-\cos\omega}{\omega}\,\cos(h\omega\nu) - \frac{\sin\omega}{\omega}\,(h\omega\nu)\cos(h\omega\nu)\right.$$
$$-\frac{1-\cos\omega}{\omega}\,(h\omega\nu)\sin(h\omega\nu) + \cos(h\omega\nu)\sum_{\gamma=1}^{N} C_\gamma(h\omega\gamma)\cos(h\omega\gamma)$$
$$\left.+\sin(h\omega\nu)\sum_{\gamma=1}^{N} C_\gamma(h\omega\gamma)\sin(h\omega\gamma)\right].$$

Now, substituting (43) into equation (29), we get the following identity with respect to $(h\nu)$

$$S_1 - S_2 + d_1\sin(h\omega\nu) + d_2\cos(h\omega\nu) = f(h\nu), \tag{44}$$

where $f(h\nu)$ is defined by (31).

Unknowns in (44) are m, n, d_1 and d_2. Equating the corresponding coefficients of $(h\omega\nu)\sin(h\omega\nu)$ and $(h\omega\nu)\cos(h\omega\nu)$ of both sides of the identity (44), for unknowns m and n we get the following system of linear equations

$$\begin{cases} m + \lambda_1^N n = \dfrac{2h\sin h\omega\ (h\omega - \sin h\omega)}{(h\omega + \sin h\omega)(h\omega\cos h\omega - \sin h\omega)}, \\ \lambda_1^N m + n\ = \dfrac{2h\sin h\omega\ (h\omega - \sin h\omega)}{(h\omega + \sin h\omega)(h\omega\cos h\omega - \sin h\omega)}, \end{cases}$$

from which

$$m = n = \frac{2h\sin h\omega\ (h\omega - \sin h\omega)}{(h\omega + \sin h\omega)(h\omega\cos h\omega - \sin h\omega)(1+\lambda_1^N)}. \tag{45}$$

The coefficients d_1 and d_2 can be found also from (44) by equating the corresponding coefficients of $\sin(h\omega\nu)$ and $\cos(h\omega\nu)$. In this way the assertion of Theorem 7 is proved. □

Proving Theorem 7 we have just solved PROBLEM 3, which is equivalent to PROBLEM 2. Thus, PROBLEM 2 is solved, i.e., the coefficients of the optimal quadrature formula (2) in the sense of Sard in the space $K_2(P_2)$ for equal spaced nodes are found.

Remark 1. Theorem 7 for $N = 2$, $\omega = 1$ gives the result of the example (h) in [31] when $[a,b] = [0,1]$.

2.5 The norm of the error functional of the optimal quadrature formula

In this subsection we calculate square of the norm of the error functional (3) for the optimal quadrature formula (2). Furthermore, we give an asymptotic analysis of this norm.

Theorem 8. *For the error functional* (3) *of the optimal quadrature formula* (2) *on the space* $K_2(P_2)$ *the following equality*

$$\|\overset{\circ}{\ell}\|^2=\frac{3\omega-\sin\omega}{2\omega^5}+\frac{h\sin\omega-\sin h\omega}{\omega^4(h\omega+\sin h\omega)}+\frac{4(1-\cos h\omega)(h-1)}{\omega^5(h\omega+\sin h\omega)h}-\frac{4\sin h\omega-2(h\omega+\sin h\omega)\cos h\omega}{\omega^5(h\omega+\sin h\omega)\sin h\omega}$$
$$+\frac{m}{2\omega^4}\left[\frac{(1-\lambda_1^2)(1-\lambda_1^N)(h\omega\cos h\omega-\sin h\omega)}{(\lambda_1^2+1-2\lambda_1\cos h\omega)\sin h\omega}-\frac{(\lambda_1+\lambda_1^{N+1})(\sin\omega+\omega)\sin h\omega+4(\lambda_1^2+\lambda_1^N)}{\lambda_1^2+1-2\lambda_1\cos h\omega}\right.$$
$$\left.-\frac{4(\lambda_1-\lambda_1^N)}{1-\lambda_1}\right],$$

holds, where λ_1 *is given in Theorem 4,* $|\lambda_1|<1$ *and m is defined by (45).*

Proof. In the equal spaced case of the nodes, using (14), we can rewrite the expression (17) in the following form

$$\|\ell\|^2=\sum_{\nu=0}^{N}C_\nu\left[\sum_{\gamma=0}^{N}C_\gamma G(h\nu-h\gamma)-f(h\nu)\right]-\sum_{\nu=0}^{N}C_\nu f(h\nu)+\frac{1}{2\omega^4}\left[2+\cos\omega-\frac{3}{\omega}\sin\omega\right],$$

where $f(h\nu)$ is defined by (31).

Hence taking into account equality (29) we get

$$\|\ell\|^2=\sum_{\nu=0}^{N}C_\nu\left(-d_1\sin(h\omega\nu)-d_2\cos(h\omega\nu)\right)$$
$$-\sum_{\nu=0}^{N}C_\nu f(h\nu)+\frac{1}{2\omega^4}\left[2+\cos\omega-\frac{3}{\omega}\sin\omega\right].$$

Using equalities (30) and (31), after some simplifications, we obtain

$$\|\ell\|^2=\frac{d_1(\cos\omega-1)-d_2\sin\omega}{\omega}-\frac{1}{4\omega^4}\left[4\sum_{\nu=0}^{N}C_\nu+\sin\omega\sum_{\nu=0}^{N}C_\nu(h\omega\nu)\cos(h\omega\nu)\right.$$
$$\left.-(1+\cos\omega)\sum_{\nu=0}^{N}C_\nu(h\omega\nu)\sin(h\omega\nu)\right]+\frac{1}{4\omega^4}\left[5+\cos\omega-\frac{2}{\omega}\sin\omega\right]. \quad (46)$$

Now from (44), equating the corresponding coefficients of $\sin(h\omega\nu)$ and $\cos(h\omega\nu)$, for d_1 and d_2 we find the following expressions

$$d_1 = \frac{\omega\cos\omega - \sin\omega}{4\omega^4} + \frac{1}{4\omega^3}\sum_{\gamma=0}^{N} C_\gamma(h\omega\gamma)\sin(h\omega\gamma) - \frac{h(h\omega - \sin h\omega)(\lambda_1^2-1)(\lambda_1^N-1)}{2\omega^3(h\omega+\sin h\omega)(1+\lambda_1^N)(\lambda_1^2+1-2\lambda_1\cos h\omega)},$$

$$d_2 = \frac{1-\cos\omega - \omega\sin\omega}{4\omega^4} + \frac{1}{4\omega^3}\sum_{\gamma=0}^{N} C_\gamma(h\omega\gamma)\cos(h\omega\gamma).$$

Substituting these expressions in (46) we get

$$\|\ell\|^2 = \frac{3\omega - \sin\omega}{2\omega^5} + \frac{h(1-\cos\omega)(h\omega-\sin h\omega)(\lambda_1^2-1)(\lambda_1^N-1)}{2\omega^4(h\omega+\sin h\omega)(1+\lambda_1^N)(\lambda_1^2+1-2\lambda_1\cos h\omega)}$$
$$+\frac{\cos\omega}{2\omega^4}\sum_{\gamma=1}^{N} C_\gamma(h\omega\gamma)\sin(h\omega\gamma) - \frac{\sin\omega}{2\omega^4}\sum_{\gamma=1}^{N} C_\gamma(h\omega\gamma)\cos(h\omega\gamma) - \frac{1}{\omega^4}\sum_{\gamma=1}^{N} C_\gamma.$$

Finally, using the expression for optimal coefficients C_γ from Theorem 7, after some calculations and simplifications, we get the assertion of Theorem 8. □

Theorem 9. *The norm of the error functional* (3) *for the optimal quadrature formula* (2) *has the form*

$$\|\overset{\circ}{\ell}\,|K_2^*(P_2)\|^2 = \frac{1}{720}h^4 + O(h^5) \quad \text{as} \quad N \to \infty. \tag{47}$$

Proof. Since

$$\lambda_1 = \frac{2h\omega - \sin(2h\omega) - 2\sin h\omega\sqrt{h^2\omega^2 - \sin^2 h\omega}}{2(h\omega\cos h\omega - \sin h\omega)} = (\sqrt{3}-2) + O(h^2)$$

and $|h\omega| < 1$, $\omega > 0$, then $|\lambda_1| < 1$ and $\lambda_1^N \to 0$ as $N \to \infty$. Thus, when $N \to \infty$ the expansion of the expression for $\|\overset{\circ}{\ell}\|^2$ (from Theorem 8) in a power series in h gives the assertion of Theorem 9. □

The next theorem gives an asymptotic optimality for our optimal quadrature formula.

Theorem 10. *Optimal quadrature formula of the form* (2) *with the error functional* (3) *in the space* $K_2(P_2)$ *is asymptotic optimal in the Sobolev space* $L_2^{(2)}(0,1)$*, i.e.,*

$$\lim_{N\to\infty} \frac{\|\overset{\circ}{\ell}\,|K_2^*(P_2)\|^2}{\|\overset{\circ}{\ell}\,|L_2^{(2)*}(0,1)\|^2} = 1. \tag{48}$$

Proof. Using Corollary 5.2 from [48] (for $m = 2$ and $\eta_0 = 0$), for square of the norm of the error functional (3) for the optimal quadrature formula (2) on the Sobolev space $L_2^{(2)}(0,1)$, we get the following expression

$$\| \overset{\circ}{\ell} | L_2^{(2)*}(0,1)\|^2 = \frac{1}{720}h^4 - \frac{h^5}{12} d \sum_{i=1}^{4} \frac{q^{N+i} + (-1)^i q}{(1-q)^{i+1}} \Delta^i 0^4 = \frac{1}{720}h^4 + O(h^5), \quad (49)$$

where d is known, $q = \sqrt{3} - 2$, $\Delta^i \gamma^4$ is the finite difference of order i of γ^4, $\Delta^i 0^4 = \Delta^i \gamma^4|_{\gamma=0}$.

Using (47) and (49) we obtain (48) and proof is finished. □

As we said in Subsection 2.1, the error (4) of the optimal quadrature formula of the form (2) in the space $K_2(P_2)$ can be estimated by the Cauchy-Schwarz inequality

$$|R_N(\varphi)| \leq \|\varphi | K_2(P_2)\| \cdot \| \overset{\circ}{\ell} | K_2^*(P_2)\|.$$

Hence taking into account Theorem 9 we get

$$|R_N(\varphi)| \leq \|\varphi | K_2(P_2)\| \left(\frac{\sqrt{5}}{60} h^2 + O(h^{5/2}) \right),$$

from which we conclude that order of the convergence of our optimal quadrature formula is $O(h^2)$.

3 Interpolation splines minimizing the semi-norm

3.1 Statement of the problem

In order to find an approximate representation of a function φ by elements of a certain finite dimensional space, it is possible to use values of this function at some points x_β, $\beta = 0, 1, \ldots, N$. The corresponding problem is called *the interpolation problem*, and the points x_β are *interpolation nodes*.

Polynomial and spline interpolations are very wide subjects in approximation theory (cf. DeVore and Lorentz [15], Mastroianni and Milovanović [34]). The theory of splines as a relatively new area has undergone a rapid progress. Many books are devoted to the theory of splines, for example, Ahlberg *et al* [1], Arcangeli *et al* [2], Attea [3], Berlinet and Thomas-Agnan [5], Bojanov *et al* [8], de Boor [14], Eubank [17], Green and Silverman [22], Ignatov and Pevniy [28], Korneichuk *et al* [29], Laurent [32], Nürnberger [38], Schumaker [44], Stechkin and Subbotin [56], Vasilenko [57], Wahba [59] and others.

If the exact values $\varphi(x_\beta)$ of an unknown function $\varphi(x)$ are known, it is usual to approximate φ by minimizing

$$\int_a^b (g^{(m)}(x))^2 dx \quad (50)$$

on the set of interpolating functions (i.e., $g(x_\beta) = \varphi(x_\beta)$, $\beta = 0,1,\dots,N$) of the Sobolev space $L_2^{(m)}(a,b)$. It turns out that the solution is the natural polynomial spline of degree $2m-1$ with knots $x_0, x_1, \dots, x_N$. It is called the interpolating D^m-spline for the points $(x_\beta, \varphi(x_\beta))$. In the non-periodic case this problem was first investigated by Holladay [27] for $m=2$, and the result of Holladay was generalized by de Boor [13] for any m. In the Sobolev space $\widetilde{L_2^{(m)}}$ of periodic functions the minimization problem of integrals of type (50) was investigated by I.J. Schoenberg [40], M. Golomb [21], W. Freeden [18, 19] and others.

In the Hilbert space $K_2(P_2)$, defined in section 1 with the semi-norm (1), we consider the following interpolation problem:

PROBLEM 5. *Find the function $S(x) \in K_2(P_2)$ which gives minimum to the semi-norm* (1) *and satisfies the interpolation condition*

$$S(x_\beta) = \varphi(x_\beta), \;\; \beta = 0,1,\dots,N,$$

for any $\varphi \in K_2(P_2)$, where $x_\beta \in [0,1]$ are the nodes of interpolation.

From [57, p.45-47] it follows that the solution $S(x)$ of PROBLEM 5 is exists, unique when $N \geq \omega$.

We give a definition of the interpolation spline function in the space $K_2(P_2)$ following [32, Chapter 4, p. 217-219].

Let $\Delta:\ 0 = x_0 < x_1 < \dots < x_N = 1$ be a mesh on the interval $[0,1]$, then the interpolation spline function with respect to Δ is a function $S(x) \in K_2(P_2)$ and satisfies the following conditions:

(i) $S(x)$ is a linear combination of functions $\sin\omega x$, $\cos\omega x$, $x\sin\omega x$ and $x\cos\omega x$ on each open mesh interval $(x_\beta, x_{\beta+1})$, $\beta = 0,1,\dots,N-1$;

(ii) $S(x)$ is a linear combination of functions $\sin\omega x$ and $\cos\omega x$ on intervals $(-\infty,0)$ and $(1,\infty)$;

(iii) $S^{(\alpha)}(x_\beta^-) = S^{(\alpha)}(x_\beta^+)$, $\alpha = 0,1,2$, $\beta = 0,1,\dots,N$;

(iv) $S(x_\beta) = \varphi(x_\beta)$, $\beta = 0,1,\dots,N$ for any $\varphi \in K_2(P_2)$.

We consider the fundamental solution $G(x)$ defined by (14) of the differential operator $\dfrac{\mathrm{d}^4}{\mathrm{d}x^4} + 2\omega^2 \dfrac{\mathrm{d}^2}{\mathrm{d}x^2} + \omega^4$.

It is clear that the third derivative of the function

$$G(x - x_\gamma) = \frac{\operatorname{sign}(x - x_\gamma)}{4\omega^3}\big(\sin(\omega x - \omega x_\gamma) - \omega(x - x_\gamma)\cos(\omega x - \omega x_\gamma)\big)$$

has a discontinuity equal to 1 at the point x_γ, and the first and the second derivatives of $G(x - x_\gamma)$ are continuous. Suppose a function $p_\gamma(x)$ coincides with the spline $S(x)$ on the interval $(x_\gamma, x_{\gamma+1})$, i.e., $p_\gamma(x) := p_{\gamma-1}(x) + C_\gamma G(x - x_\gamma)$, $x \in (x_\gamma, x_{\gamma+1})$, where C_γ is the jump of the function $S'''(x)$ at x_γ:

$$C_\gamma = S'''(x_\gamma^+) - S'''(x_\gamma^-).$$

Then the spline $S(x)$ can be written in the following form

$$S(x)=\sum_{\gamma=0}^{N} C_\gamma G(x-x_\gamma)+p_{-1}(x), \tag{51}$$

where $p_{-1}(x)=d_1\sin\omega x+d_2\cos\omega x$, with d_1, d_2 real numbers.

Furthermore, the function $S(x)$ satisfies the condition (ii) if the function $\frac{1}{4\omega^3}\sum_{\gamma=0}^{N} C_\gamma[\sin(\omega x-\omega x_\gamma)-\omega(x-x_\gamma)\cos(\omega x-\omega x_\gamma)]$ is a linear combination of the functions $\sin\omega x$ and $\cos\omega x$. Hence we get the following conditions for C_γ

$$\sum_{\gamma=0}^{N} C_\gamma \sin(\omega x_\gamma)=0,\quad \sum_{\gamma=0}^{N} C_\gamma \cos(\omega x_\gamma)=0.$$

Taking into account the last two equations and the interpolation condition (iv) for the coefficients C_γ, $\gamma=0,1,2,\ldots,N$, d_1 and d_2 of spline (51) we obtain the following system of $N+3$ linear equations

$$\sum_{\gamma=0}^{N} C_\gamma G(x_\beta-x_\gamma)+d_1\sin(\omega x_\beta)+d_2\cos(\omega x_\beta)=\varphi(x_\beta),\ \ \beta=0,1,\ldots,N, \tag{52}$$

$$\sum_{\gamma=0}^{N} C_\gamma \sin(\omega x_\gamma)=0, \tag{53}$$

$$\sum_{\gamma=0}^{N} C_\gamma \cos(\omega x_\gamma)=0, \tag{54}$$

where $\varphi\in K_2(P_2)$.

Note that the analytic representation (51) of the interpolation spline $S(x)$ and the system (52)-(54) for the coefficients can be also obtained from [57, p. 45-47, Theorem 2.2].

It should be noted that systems for the coefficients of D^m-splines similar to the system (52)–(54) were investigated, for example, in [2, 16, 28, 32, 57].

In the work [49], using S.L. Sobolev's method, it was constructed the interpolation splines minimizing the semi-norm in the space $W_2^{(m,m-1)}(0,1)$, where $W_2^{(m,m-1)}(0,1)$ is the space of functions φ which $\varphi^{(m-1)}$ is absolutely continuous and $\varphi^{(m)}$ belongs to $L_2(0,1)$ and $\int_0^1(\varphi^{(m)}(x)+\varphi^{(m-1)}(x))^2dx<\infty$.

The main aim of the present section is to solve PROBLEM 5, i.e., to solve the system (52)–(54) for equally spaced nodes $x_\beta=h\beta$, $\beta=0,1,\ldots,N$, $h=1/N$, $N\geq\omega>0$ and to find analytic formulas for coefficients C_γ, $\gamma=0,1,\ldots,N$, d_1 and d_2 of $S(x)$.

3.2 An algorithm for computing the coefficients of interpolation splines

In this subsection we give an algorithm for solving the system of equations (52)–(54), when the nodes x_β are equally spaced. Here we use similar method proposed by S.L. Sobolev [53, 55] for finding the coefficients of optimal quadrature formulas in the space $L_2^{(m)}$.

Here also we use the concept of discrete argument functions and operations on them (see subsection 2.4).

Suppose that $C_\beta = 0$ when $\beta < 0$ and $\beta > N$. Using Definition 3, we write equalities (52)–(54) as follows:

$$G(h\beta) * C_\beta + d_1 \sin(h\omega\beta) + d_2 \cos(h\omega\beta) = \varphi(h\beta), \ \ \beta = 0,1,\ldots,N, \quad (55)$$

$$\sum_{\beta=0}^{N} C_\beta \sin(h\omega\beta) = 0, \quad (56)$$

$$\sum_{\beta=0}^{N} C_\beta \cos(h\omega\beta) = 0, \quad (57)$$

where $G(h\beta)$ is the function of discrete argument corresponding to the function G given in (14).

Thus we have the following problem.

PROBLEM 6. *Find the coefficients C_β, $\beta = 0,1,\ldots,N$, and the constants d_1, d_2 which satisfy the system* (55)–(57).

Further we investigate PROBLEM 6 which is equivalent to PROBLEM 5. Namely, instead of C_β we introduce the following functions

$$v(h\beta) = G(h\beta) * C_\beta, \quad (58)$$

$$u(h\beta) = v(h\beta) + d_1 \sin(h\omega\beta) + d_2 \cos(h\omega\beta). \quad (59)$$

In such statement it is necessary to express the coefficients C_β by the function $u(h\beta)$. For this we use the operator $D(h\beta)$ which is given in Theorem 4.

Then, taking into account (59) and Theorems 4, 5, for the coefficients C_β of the spline $S(x)$ we have

$$C_\beta = D(h\beta) * u(h\beta). \quad (60)$$

Thus if we find the function $u(h\beta)$ then the coefficients C_β can be obtained from equality (60). In order to calculate the convolution (60) we need a representation of the function $u(h\beta)$ for all integer values of β. From equality (55) we get that $u(h\beta) = \varphi(h\beta)$ when $h\beta \in [0,1]$. Now we need to find a representation of the function $u(h\beta)$ when $\beta < 0$ and $\beta > N$.

Since $C_\beta = 0$ when $h\beta \notin [0,1]$ then $C_\beta = D(h\beta) * u(h\beta) = 0$, $h\beta \notin [0,1]$. Now we calculate the convolution $v(h\beta) = G(h\beta) * C_\beta$ when $\beta \le 0$ and $\beta \ge N$.

Suppose $\beta \le 0$ then taking into account equalities (58), (56), (57), we have

$$
\begin{aligned}
v(h\beta) &= \sum_{\gamma=-\infty}^{\infty} C_\gamma G(h\beta - h\gamma)\\
&= \sum_{\gamma=0}^{N} C_\gamma \frac{\mathrm{sign}(h\beta-h\gamma)}{4\omega^3}\Big(\sin(h\omega\beta-h\omega\gamma)-(h\omega\beta-h\omega\gamma)\cos(h\omega\beta-h\omega\gamma)\Big)\\
&= -\frac{1}{4\omega^3}\sum_{\gamma=0}^{N} C_\gamma \Big\{\sin(h\omega\beta)\cos(h\omega\gamma)-\cos(h\omega\beta)\sin(h\omega\gamma)\\
&\quad -(h\omega\beta)\Big[\cos(h\omega\beta)\cos(h\omega\gamma)+\sin(h\omega\beta)\sin(h\omega\gamma)\Big]\\
&\quad +(h\omega\gamma)\Big[\cos(h\omega\beta)\cos(h\omega\gamma)+\sin(h\omega\beta)\sin(h\omega\gamma)\Big]\Big\}\\
&= -\frac{1}{4\omega^3}\cos(h\omega\beta)\sum_{\gamma=0}^{N} C_\gamma (h\omega\gamma)\cos(h\omega\gamma)-\frac{1}{4\omega^3}\sin(h\omega\beta)\sum_{\gamma=0}^{N} C_\gamma(h\omega\gamma)\sin(h\omega\gamma).
\end{aligned}
$$

Denoting $b_1 = \frac{1}{4\omega^3}\sum_{\gamma=0}^{N} C_\gamma(h\omega\gamma)\sin(h\omega\gamma)$ and $b_2 = \frac{1}{4\omega^3}\sum_{\gamma=0}^{N} C_\gamma(h\omega\gamma)\cos(h\omega\gamma)$ we get for $\beta \le 0$

$$v(h\beta) = -b_1\sin(h\omega\beta) - b_2\cos(h\omega\beta).$$

and for $\beta \ge N$

$$v(h\beta) = b_1\sin(h\omega\beta) + b_2\cos(h\omega\beta).$$

Now, setting

$$d_1^- = d_1 - b_1,\ d_2^- = d_2 - b_2,\ d_1^+ = d_1 + b_1,\ d_2^+ = d_2 + b_2$$

we formulate the following problem:

PROBLEM 7. *Find the solution of the equation*

$$D(h\beta) * u(h\beta) = 0, \quad h\beta \notin [0,1], \tag{61}$$

in the form:

$$u(h\beta) = \begin{cases} d_1^-\sin(h\omega\beta) + d_2^-\cos(h\omega\beta), & \beta \le 0,\\ \varphi(h\beta), & 0 \le \beta \le N,\\ d_1^+\sin(h\omega\beta) + d_2^+\cos(h\omega\beta), & \beta \ge N. \end{cases} \tag{62}$$

where d_1^-, d_2^-, d_1^+, d_2^+ are unknown coefficients.

It is clear that

$$
\begin{aligned}
d_1 &= \frac{1}{2}\left(d_1^+ + d_1^-\right),\ \ d_2 = \frac{1}{2}\left(d_2^+ + d_2^-\right),\\
b_1 &= \frac{1}{2}\left(d_1^+ - d_1^-\right),\ \ b_2 = \frac{1}{2}\left(d_2^+ - d_2^-\right).
\end{aligned}
\tag{63}
$$

These unknowns $d_1^-, d_2^-, d_1^+, d_2^+$ can be found from equation (61), using the function $D(h\beta)$. Then the explicit form of the function $u(h\beta)$ and coefficients C_β, d_1, d_2 can be found. Thus, PROBLEM 7 and respectively PROBLEMS 6 and 5 can be solved.

In the next subsection we realize this algorithm for computing the coefficients C_β, $\beta = 0, 1, \dots, N$, d_1 and d_2 of the interpolation spline (51).

3.3 Computation of coefficients of the interpolation spline

In this subsection using the algorithm from the previous subsection we obtain explicit formulae for coefficients of the interpolation spline (51) which is the solution of PROBLEM 5.

It should be noted that the interpolation spline (51), which is the solution of PROBLEM 5, is exact for the functions $\sin \omega x$ and $\cos \omega x$.

The following holds:

Theorem 11. *Coefficients of interpolation spline* (51) *which minimizes the seminorm* (1) *with equally spaced nodes in the space* $K_2(P_2)$ *have the following form:*

$$
\begin{aligned}
C_0 &= Cp\varphi(0) + p[\varphi(h) - d_1^- \sin(h\omega) + d_2^- \cos(h\omega)] \\
&\quad + \frac{A_1 p}{\lambda_1}\left[\sum_{\gamma=0}^{N} \lambda_1^{\gamma}\varphi(h\gamma) + M_1 + \lambda_1^N N_1\right] \\
C_\beta &= Cp\varphi(h\beta) + p[\varphi(h(\beta-1)) + \varphi(h(\beta+1))] \\
&\quad + \frac{A_1 p}{\lambda_1}\left[\sum_{\gamma=0}^{N} \lambda_1^{|\beta-\gamma|}\varphi(h\gamma) + \lambda_1^{\beta} M_1 + \lambda_1^{N-\beta} N_1\right], \quad \beta = 1, 2, \dots, N-1, \\
C_N &= Cp\varphi(1) + p[\varphi(h(N-1)) + d_1^+ \sin(\omega + h\omega) + d_2^+ \cos(\omega + h\omega)] \\
&\quad + \frac{A_1 p}{\lambda_1}\left[\sum_{\gamma=0}^{N} \lambda_1^{N-\gamma}\varphi(h\gamma) + \lambda_1^N M_1 + N_1\right], \\
d_1 &= \frac{1}{2}(d_1^+ + d_1^-), \qquad d_2 = \frac{1}{2}(d_2^+ + d_2^-),
\end{aligned}
$$

where p, A_1, C *and* λ_1 *are defined by* (35), (36),

$$
M_1 = \frac{\lambda_1[d_2^-(\cos(h\omega) - \lambda_1) - d_1^- \sin(h\omega)]}{\lambda_1^2 + 1 - 2\lambda_1 \cos(h\omega)}, \tag{64}
$$

$$
N_1 = \frac{\lambda_1[d_2^+(\cos(\omega + h\omega) - \lambda_1 \cos\omega) + d_1^+(\sin(\omega + h\omega) - \lambda_1 \sin\omega)]}{\lambda_1^2 + 1 - 2\lambda_1 \cos(h\omega)}, \tag{65}
$$

and d_1^+, d_1^-, d_2^+, d_2^- *are defined by* (66) *and* (72).

Proof. First we find the expressions for d_2^- and d_2^+. From (62), when $\beta = 0$ and $\beta = N$ we get

$$d_2^- = \varphi(0), \quad d_2^+ = \frac{\varphi(1)}{\cos\omega} - d_1^+ \tan\omega. \tag{66}$$

Now we have two unknowns d_1^- and d_1^+. These unknowns we find from (61) when $\beta = -1$ and $\beta = N+1$.

Taking into account (62) and Definition 3 from (61) we have

$$\sum_{\gamma=-\infty}^{-1} D(h\beta - h\gamma)[d_1^- \sin(h\omega\gamma) + d_2^- \cos(h\omega\gamma)] + \sum_{\gamma=0}^{N} D(h\beta - h\gamma)\varphi(h\gamma)$$
$$+ \sum_{\gamma=N+1}^{\infty} D(h\beta - h\gamma)[d_1^+ \sin(h\omega\gamma) + d_2^+ \cos(h\omega\gamma)] = 0,$$

where $\beta < 0$ and $\beta > N$.

Hence for $\beta = -1$ and $\beta = N+1$ we get the following system for d_1^-, d_1^+, d_2^-, d_2^+:

$$\begin{aligned} -d_1^- \sum_{\gamma=1}^{\infty} D(h\gamma - h)\sin(h\omega\gamma) &+ d_2^- \sum_{\gamma=1}^{\infty} D(h\gamma - h)\cos(h\omega\gamma) \\ &+ d_1^+ \sum_{\gamma=1}^{\infty} D(h(N+\gamma)+h)\sin(\omega + h\omega\gamma) \\ &+ d_2^+ \sum_{\gamma=1}^{\infty} D(h(N+\gamma)+h)\cos(\omega + h\omega\gamma) \\ &= -\sum_{\gamma=0}^{N} D(h\gamma + h)\varphi(h\gamma), \end{aligned} \tag{67}$$

$$\begin{aligned} -d_1^- \sum_{\gamma=1}^{\infty} D(h(N+\gamma)+h)\sin(h\omega\gamma) &+ d_2^- \sum_{\gamma=1}^{\infty} D(h(N+\gamma)+h)\cos(h\omega\gamma) \\ &+ d_1^+ \sum_{\gamma=1}^{\infty} D(h\gamma - h)\sin(\omega + h\omega\gamma) \\ &+ d_2^+ \sum_{\gamma=1}^{\infty} D(h\gamma - h)\cos(\omega + h\omega\gamma) \\ &= -\sum_{\gamma=0}^{N} D(h(N+1) - h\gamma)\varphi(h\gamma). \end{aligned} \tag{68}$$

Since $|\lambda_1| < 1$, the series in the system (67), (68) are convergent.

Using (66) and taking into account (34), after some calculations and simplifications, from (67), (68) we obtain

$$B_{11}d_1^- + B_{12}d_1^+ = T_1, \qquad B_{21}d_1^- + B_{22}d_1^+ = T_2,$$

where

$$B_{11}=\lambda_1\sin(h\omega),\ B_{12}=-\frac{\lambda_1^{N+1}\sin(h\omega)}{\cos\omega},\ B_{21}=\lambda_1^{N+1}\sin(h\omega),\ B_{22}=-\frac{\lambda_1\sin(h\omega)}{\cos\omega}, \quad (69)$$

$$T_1=\frac{2h\omega\lambda_1\sin^2(h\omega)}{h\omega\cos(h\omega)-\sin(h\omega)}\sum_{\gamma=0}^{N}\lambda_1^{\gamma}\varphi(h\gamma) +(\lambda_1\cos(h\omega)-1)\varphi(0)+\lambda_1^{N+1}(\cos(h\omega)-\lambda_1-\tan\omega\sin(h\omega))\varphi(1), \quad (70)$$

$$T_2=\frac{2h\omega\lambda_1\sin^2(h\omega)}{h\omega\cos(h\omega)-\sin(h\omega)}\sum_{\gamma=0}^{N}\lambda_1^{N-\gamma}\varphi(h\gamma) +\lambda_1^{N+1}(\cos(h\omega)-\lambda_1)\varphi(0)+(\lambda_1\cos(h\omega)-1-\lambda_1\tan\omega\sin(h\omega))\varphi(1). \quad (71)$$

Hence we get

$$d_1^-=\frac{T_1B_{22}-T_2B_{12}}{B_{11}B_{22}-B_{12}B_{21}},\quad d_1^+=\frac{T_2B_{11}-T_1B_{21}}{B_{11}B_{22}-B_{12}B_{21}}, \quad (72)$$

where $B_{11},B_{12},B_{21},B_{22},T_1$, and T_2 are defined by (69) - (71)

Combining (63), (66) and (72) we obtain d_1 and d_2 which are given in the statement of Theorem 11.

Now, we calculate the coefficients C_β, $\beta=0,1,\dots,N$. Taking into account (62) from (60) for C_β we have

$$\begin{aligned}C_\beta &= D(h\beta)*u(h\beta)\\ &=\sum_{\gamma=-\infty}^{\infty}D(h\beta-h\gamma)u(h\gamma)\\ &=\sum_{\gamma=1}^{\infty}D(h\beta+h\gamma)[-d_1^-\sin(h\omega\gamma)+d_2^-\cos(h\omega\gamma)]+\sum_{\gamma=0}^{N}D(h\beta-h\gamma)\varphi(h\gamma)\\ &\quad+\sum_{\gamma=1}^{\infty}D(h(N+\gamma)-h\beta)[d_1^+\sin(\omega+h\omega\gamma)+d_2^+\cos(\omega+h\omega\gamma)],\end{aligned}$$

from which, using (34) and taking into account notations (64), (65), when $\beta=0,1,\dots,N$, for C_β we get expressions from the statement of Theorem 11.

Remark 2. It should be noted that from the results of this section when $\omega=1$ we get the results of [26].

3.4 Numerical results

In this subsection, in numerical examples, we compare the interpolation spline (51) with the natural cubic spline (D^2-spline).

It is known that (see, for instance, [2, 13, 16, 27, 28, 32, 57]) the natural cubic spline minimize the integral $\int_0^1 (\varphi''(x))^2 dx$ in the Sobolev space $L_2^{(2)}(0,1)$ of functions with a square integrable 2-nd generalized derivative. For convenience we denote the natural cubic spline as $Scubic(x)$. In numerical examples we use standard function "spline(X,Y,x,cubic)" of Maple package for the natural cubic spline.

Here first we consider the case $\omega = 1$ and give the numerical results which were given in [26].

We apply the interpolation spline (51) and the natural cubic spline to approximation of the functions

$$f_1(x) = \mathrm{e}^x, \quad f_2(x) = \tan x, \quad f_3(x) = \frac{313x^4 - 6900x^2 + 15120}{13x^4 + 660x^2 + 15120}.$$

Using Theorem 11 and the standard Maple function "spline(X,Y,x,cubic)", with $N = 5$ and $N = 10$, we get the corresponding interpolation splines denoted by $S_N(f_k;x)$, $k = 1,2,3$ for the interpolation spline (51) and $Scubic_N(f_k;x)$, $k = 1,2,3$ for the natural cubic spline.

The corresponding absolute errors $|f_k(x) - S_N(f_k;x)|$ and $|f_k(x) - Scubic_N(f_k;x)|$ on $[0,1]$, for $k = 1,2,$ and 3 are displayed in Figures 1, 3, 5, and 2, 4, 6 respectively.

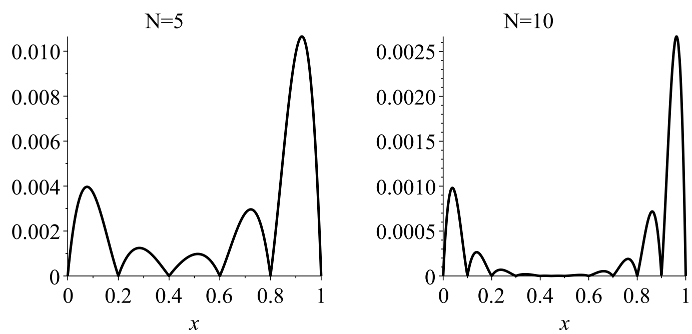

Fig. 1 Graphs of absolute errors $|f_1(x) - S_N(f_1;x)|$ for $N = 5$ and $N = 10$

As we can see the smallest errors in these cases appear in the Figure 5, because $f_3(x)$ is a rational approximation for the function $\cos x$ (cf. [23, p. 66]), and the interpolation spline (51) is exact for the trigonometric functions $\sin x$ and $\cos x$.

It should be noted that we used the same functions $f_k(x)$, $k = 1,2,3$, to test an optimal quadrature formula in the sense of Sard in the space $K_2(P_2)$,

$$I(\varphi) := \int_0^1 \varphi(x)\,\mathrm{d}x \cong \sum_{\nu=0}^{N} C_\nu \varphi(x_\nu) =: Q_N(\varphi), \tag{73}$$

which have been constructed recently in our paper [25]. The weight coefficients in (73) are

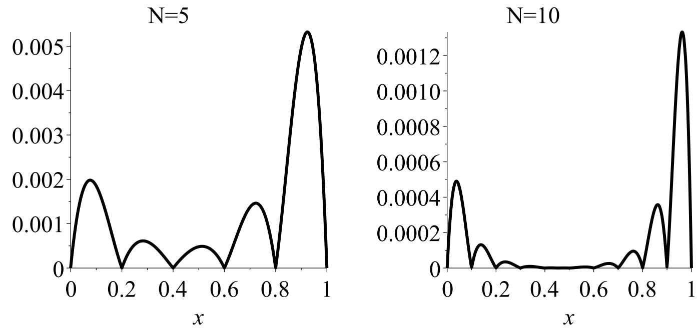

Fig. 2 Graphs of absolute errors $|f_1(x) - Scubic_N(f_1;x)|$ for $N=5$ and $N=10$

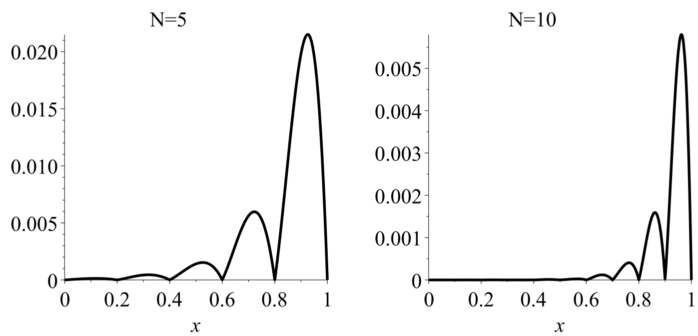

Fig. 3 Graphs of absolute errors $|f_2(x) - S_N(f_2;x)|$ for $N=5$ and $N=10$

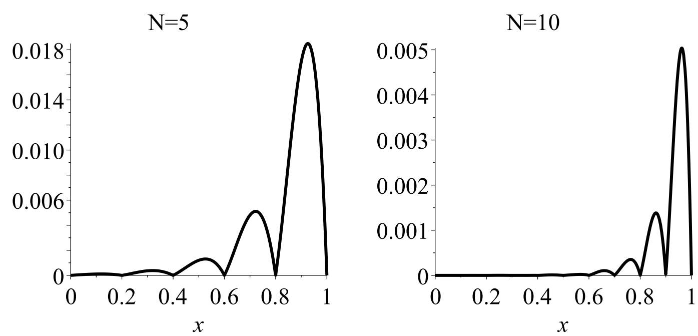

Fig. 4 Graphs of absolute errors $|f_2(x) - Scubic_N(f_2;x)|$ for $N=5$ and $N=10$

$$C_0 = C_N = \frac{2\sin h - (h+\sin h)\cos h}{(h+\sin h)\sin h} + \frac{h - \sin h}{(h+\sin h)\sin h(1+\lambda_1^N)}\left(\lambda_1 + \lambda_1^{N-1}\right)$$

and

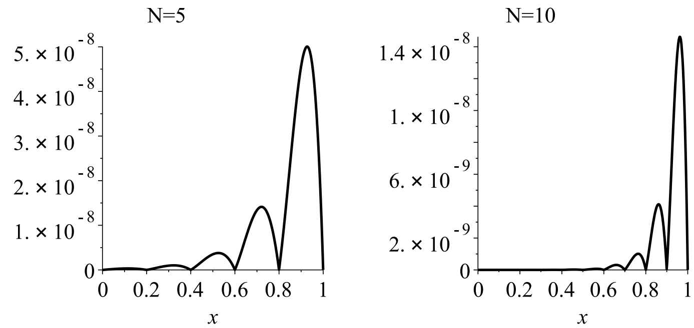

Fig. 5 Graphs of absolute errors $|f_3(x) - S_N(f_3;x)|$ for $N = 5$ and $N = 10$

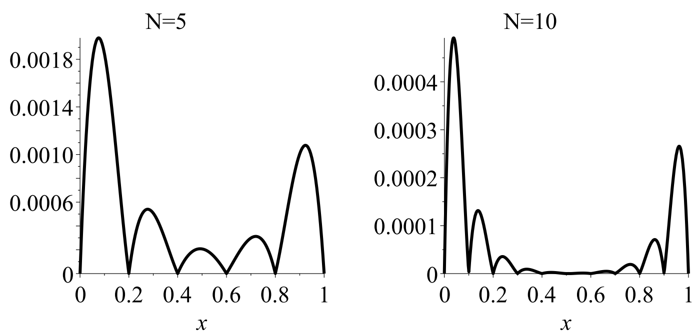

Fig. 6 Graphs of absolute errors $|f_3(x) - Scubic_N(f_3;x)|$ for $N = 5$ and $N = 10$

$$C_\nu = \frac{4(1-\cos h)}{h+\sin h} + \frac{2h(h-\sin h)\sin h}{(h+\sin h)(h\cos h - \sin h)(1+\lambda_1^N)}\left(\lambda_1^\nu + \lambda_1^{N-\nu}\right),$$

for $\nu = 1, \ldots, N-1$, where λ_1 is given as in (36), with $\omega = 1$ and $|\lambda_1| < 1$.

In [25] we have obtained the approximate numerical values $Q_N(f_k) = \sum_{\nu=0}^{N} C_\nu f_k(x_\nu)$ for the corresponding integrals $I(f_k)$, $k = 1,2,3$, taking $N = 10, 100$, and 1000. These approximate values we also obtain if we integrate the corresponding interpolation splines $S_N(f_k;x)$ over $[0,1]$, i.e., $Q_N(f_k) = I(S_N(f_k;x))$.

Now we consider the values of the difference $|S_N(f_1;x) - Scubic_N(f_1;x)|$ *for the cases* $\omega = 10,\ 1,\ 0.1,\ 0.01$*, with* $N = 10$*.*

From the Figures 7 and 8 we can see that $S_N(f_1;x)$ tends to $Scubic_N(f_1;x)$ as $\omega \to 0$.

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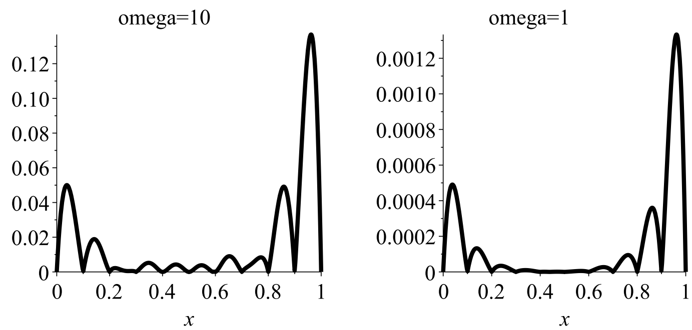

Fig. 7 Graphs of absolute errors $|S_N(f_1;x) - Scubic_N(f_1;x)|$ for $\omega = 10, 1$ and $N = 10$

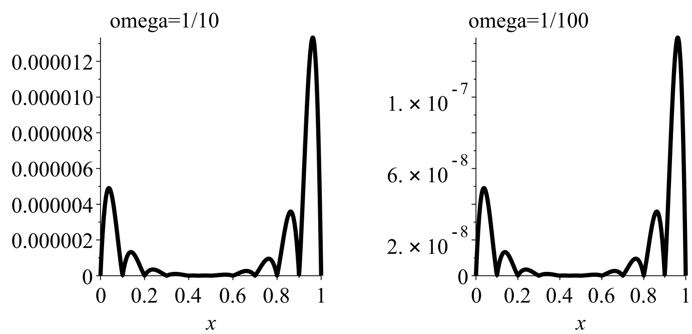

Fig. 8 Graphs of absolute errors $|S_N(f_1;x) - Scubic_N(f_1;x)|$ for $\omega = 0.1, 0.01$ and $N = 10$

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